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Multi-wave interaction theory for wind-generated surface gravity waves

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Consistently employing the assumption of localness of wave–wave interactions in the wavenumber space, the Kolmogorov treatment of the energy cascade is applied to the case of wind-generated surface gravity waves. The effective number ν of resonantly interacting wave harmonics is not limited to four but is found as a solution of a coupled system of equations expressing: (i) the dependence of the spectrum shape on the degree of the wave nonlinearity, and (ii) the continuity of the wave action flux through the spectrum given a continuous positive input from wind. The latter is specified in a Miles-type fashion, and a simple scaling relationship based on the concept of the turnover time is derived in place of the kinetic equation. The mathematical problem is reduced to an ordinary differential equation of first order. The exponent in the 'power law' for the spectral density of the wave potential energy and the effective number of resonantly interacting wave harmonics are found as functions of the wave frequency and of external factors of wind–wave interaction. The solution is close to the Zakharov–Filonenko spectrum at low frequencies and low wind input while approaching the Phillips spectrum at high frequencies and sufficiently high wind.

1. Introduction

Owing to an intrinsically small variance of the sea surface slope, problems of wave turbulence are successfully (and most naturally) treated by small-perturbation techniques. An important accomplishment is the kinetic equation derived in the approximation of four-wave interactions (Hasselmann 1962; Zakharov & Filonenko 1966). Mathematical complexity of higher-order terms in practice confines perturbation theories to accounting for only the first two terms of the interaction Hamiltonian, $H_{\text{int}} = H_1 + H_2 + \dots$ (Zakharov 1984). Analysis of five-wave interactions (term H_3) requires extensive numerical effort (e.g. West *et al.* 1987), while the inclusion of H_3 into a statistical description of wave dynamics (the kinetic equation) has not even been attempted.

The goal of the present work is to estimate effects of higher-order resonant wave–wave interactions on the high-frequency range of the wave spectrum using a scaling approach. The central idea is based on the Kolmogorov argument concerned with the cascade nature of the energy transfer down the spectrum and the assumption of localness of the energy exchange between individual Fourier components (Kolmogorov 1941). This idea was followed by, for example, Kraichnan (1972) and Frisch, Sulem & Nelkin (1978) who studied the fluid turbulence and by Zakharov & L'vov (1975), Kitaigorodskii (1983), Larraza (1987) and Larraza, Garrett & Putterman (1990) who exploited it in the analysis of purely inertial spectra of surface gravity waves exchanging energy, action and momentum within resonant wave tetrads. For four-wave interactions the localness hypothesis can be validated by direct calculations (Zakharov 1984). The new element introduced in the present work is the step in which the effective number, ν, of the resonantly interacting components is taken to be an unknown function of the problem. This number is then determined as a function of the external conditions, i.e. the spectral density of the energy input, $q(\boldsymbol{k})$. Although this approach obviates many obstacles of perturbation theories, it cannot be rigorously justified. However, some heuristic arguments suggested below give it some weight.

The four-wave approximation breaks down either at high wavenumbers – owing to the statistical self-affinity of the surface elevation field resulting in the growth of the local wave steepness ka_k (Glazman & Weichman 1989) – or/and at high winds when the energy input exceeds a certain threshold. As ka_k increases, the relative importance of H_3 and other high-order terms in $H_{\rm int}$ grows. Correspondingly, the number, ν, of the resonantly interacting harmonics contributing to the collision integral must increase as well. Larraza (1987) offered an instructive analogy with phonons in solids (see also Klemens 1965): the number of phonons increases with increasing internal energy of the solid. The growth of ν manifests itself via an intermittent occurrence of steep and breaking wavelets whose local nonlinearity, hence the number of Fourier terms required to represent a near-cusp shape, is very high. Since such events are statistically rare, the overall wave dynamics remain dominated by a weakly nonlinear inertial cascade. However, because the probability of such wavelets is finite (Glazman 1986; Glazman & Weichman 1989), a heuristic notion of the mean 'effective' value of ν appears appropriate. Another manifestation of a gradual growth of the mean ν is an accelerated (as compared to (1.3)) rate of spectral roll-off with an increasing wavenumber and frequency (Forristall 1981). This roll-off has been represented (Glazman, Pihos & Ip 1988; Glazman & Weichman 1989; Glazman & Srokosz 1991) by a 'generalized spectrum'

$$S(\omega) = \alpha\rho g^3(U/g)^{4\mu}\omega^{-5+4\mu} \tag{1.1}$$

in which μ can change from zero to about $\frac{1}{4}$, or even greater, as the sea maturity increases from that of poorly developed seas dominated by the Phillips spectrum (1.2) to that of moderately and well-developed seas dominated by the Zakharov–Filonenko (1.3) or Zakharov–Zaslavskii (2.10) spectra, respectively. However, such an *ad hoc* approach does not contribute to our understanding of the underlying physical mechanisms and of the characteristic wavenumbers/frequencies at which the accelerated roll-off should occur.

The limiting rate of the spectral roll-off attained at $\nu \rightarrow \infty$ corresponds to the Phillips (1958) spectrum for a highly nonlinear wave field:

$$S(\omega) = B\rho g^3\omega^{-5}, \tag{1.2}$$

where B is known as the universal Phillips constant, and $S(\omega)$ represents the spectral density of the wave potential energy per unit surface area. The quadruplet wave–wave interactions yield a relatively slow rate of the spectrum roll-off (Zakharov & Filonenko 1966):

$$S(\omega) = \alpha_q(\rho g^3)^{\frac{2}{3}}Q^{\frac{1}{3}}\omega^{-4}, \tag{1.3}$$

where α_q is a Kolmogorov constant and Q is the (constant) surface density of the energy flux down the spectrum. The spectra (1.2) and (1.3) represent two limiting cases and correspond to two dramatically different regimes of surface-height spatial variations (Glazman & Weichman 1989). The transition between these two regimes is of great intrinsic interest and has important implications for the interpretation of microwave remote sensing signatures of the ocean surface (Glazman 1990, 1991; Glazman & Srokosz 1991) and for air–sea interactions.

2. The constant flux of energy and wave action: cascade models

We start with a conservative inertial cascade when all the energy is supplied at low wavenumbers, k_0. As usual (Frisch *et al.* 1978), we discretize the continuous spectrum $S(\omega)$ by introducing the energy that is transferred to higher and higher wavenumbers k_n (implying a cascade process whereby the lengthscale $l_n = 1/k_n$ is a fixed fraction of l_{n-1}, for instance $k_n = 2^n k_0$) by nonlinear wave–wave interactions between ν resonant wavenumbers. Here, n labels all relevant quantities related to a given step in the cascade. On the nth step the energy is

$$E_n = \int_{\omega_n}^{\omega_{n+1}} S(\omega)\,\mathrm{d}\omega \quad \left(\equiv \int_{k_n}^{k_{n+1}} E(k)\,\mathrm{d}k\right). \tag{2.1}$$

$E(k)$ is the spectral density of the wave potential energy related to the two-dimensional wavenumber spectrum $F(k, \Theta)$ by

$$E(k) = \int_{-\pi}^{\pi} F(k, \Theta)\, k\,\mathrm{d}\Theta.$$

The wave amplitude a_n on the nth step is found from

$$E_n = \rho g a_n^2. \tag{2.2}$$

Based on the scaling of the collision integral $\nabla_k T(\boldsymbol{k})$ in the four-wave approximation, the characteristic time (the 'turnover time') for the energy exchange at step n can be related to the other characteristic scales of the problem. Let us write this result in the form used by Larraza *et al.* (1990):

$$t_n^{-1} \approx \omega_n (k_n a_n)^4. \tag{2.3}$$

Each additional Fourier component entering the interaction corresponds to a factor $(k_n a_n)^2$ to be included into the turnover time. The ν-wave interaction ($\nu \geqslant 4$) yields (Larraza 1987; Larraza *et al.* 1990):

$$t_n^{-1} \approx \omega_n (k_n a_n)^{2(\nu-2)}. \tag{2.4}$$

The rate of energy flux through the spectrum is related to the turnover time by

$$E_n / t_n \approx Q. \tag{2.5}$$

Equations (2.5), (2.4) and (2.2) yield

$$E_n \approx Q^{1/(\nu-1)} (\rho g^3)^{(\nu-2)/(\nu-1)} \omega^{(7-4\nu)/(\nu-1)}, \tag{2.6}$$

where an approximate dispersion relation

$$\omega_n^2 \approx k_n g \tag{2.7}$$

has been used. The continuous spectrum,

$$S(\omega) \sim E_n/\omega, \tag{2.8}$$

takes the form:

$$S(\omega) = \alpha Q^{1/(\nu-1)}(\rho g^3)^{(\nu-2)/(\nu-1)}\omega^{(8-5\nu)/(\nu-1)}, \tag{2.9}$$

where the Kolmogorov constant, α, enables one to write (2.9) as an equality. For four-wave conservative interactions, the Kolmogorov constant can be estimated (Zakharov 1984) by calculating the collisional integral. For $\nu = 4$ and $\nu \to \infty$ equation (2.9) reduces to (1.2) and (1.3), respectively. As ν increases, the turnover time also increases and the influence of Q in (2.9) becomes ever less important. It disappears completely in the limit of $\nu \to \infty$ (the Phillips spectrum), when the energy transfer to viscous scales is due largely to the breaking of sufficiently steep gravity waves rather than to the continuous inertial cascade.

It is easy to verify that the same approach allows one to derive the Zakharov–Zaslavskii (1982) spectrum for the inverse cascade:

$$S(\omega) = \alpha_P(\rho g^3)^{\frac{2}{3}}P^{\frac{1}{3}}\omega^{-\frac{11}{3}}. \tag{2.10}$$

Since this spectrum is based on the conservation of the wave action flux, P, equation (2.5) must be replaced with $N_n/t_n \approx P$, where

$$N_n = \rho g a_n^2\, \omega_n^{-1} \tag{2.11}$$

is the wave action. The Kolmogorov constant α_P is, generally, different from that appearing in (1.3) and (2.9). Unlike the energy flux, P is conserved in the cascade only in the approximation of four-wave interactions. Hence, the equation $N_n/t_n \approx P$ can be applied only with $\nu = 4$. The inverse cascade will not be considered in this paper, hence the theory presented below is relevant to the high-frequency range dominated by the direct cascade.

3. Continuous distribution in (k, ω) of the wind input

In the stationary case ($\partial N/\partial t = 0$), the kinetic equation for the frequency range well above the spectrum peak frequency can be written in the form (Phillips 1977)

$$\nabla_k \cdot \boldsymbol{T}(\boldsymbol{k}) = \boldsymbol{p}(\boldsymbol{k}). \tag{3.1}$$

Here, $\nabla_k \cdot \boldsymbol{T}(\boldsymbol{k})$ is the collision integral which represents the spectral density of the action flux due to nonlinear wave–wave interaction and $\boldsymbol{p}(\boldsymbol{k})$ is the input spectral flux of action. In addition to the positive input, p^+, from wind, this flux may contain a negative component, p^-, due to the high-frequency dissipation. For simplicity, this component is not explicitly treated in the present work, and $\boldsymbol{p}(\boldsymbol{k})$ is assumed to be continuous and positive. The action balance for a given step of the cascade is found by integrating (3.1) over the corresponding wavenumber range:

$$\int_{k_n}^{k_{n+1}} \nabla_k T(k)\,\mathrm{d}k = P_n, \quad \text{where} \quad P_n = \int_{k_n}^{k_{n+1}} p(k)\,\mathrm{d}k. \tag{3.2}$$

In (3.2) and hereinafter, the bold symbols in $\nabla_k \cdot \boldsymbol{T}(\boldsymbol{k})$ and $\boldsymbol{p}(\boldsymbol{k})$ are replaced by regular ones to denote that the angular integration has been carried out as follows:

$$\nabla_k T(k) \equiv \int_{-\pi}^{\pi} \nabla_k \cdot \boldsymbol{T}(\boldsymbol{k})\, k\,\mathrm{d}\Theta.$$

The energy balance is given by

$$\int_{k_n}^{k_{n+1}} \nabla_k T(k)\,\omega\,\mathrm{d}k = Q_n, \quad \text{where} \quad Q_n = \int_{k_n}^{k_{n+1}} p(k)\,\omega\,\mathrm{d}k. \tag{3.3}$$

The fact that the turnover time t_n has been found through the scaling of the collision integral allows one to approximate the left-hand side of (3.2) as

$$\int_{k_n}^{k_{n+1}} \nabla_k T(k)\,\mathrm{d}k \approx N_n/t_n. \tag{3.4}$$

Similarly, the left-hand side of (3.3) can be approximated as

$$\int_{k_n}^{k_{n+1}} \nabla_k T(k)\,\omega\,\mathrm{d}k \approx E_n/t_n. \tag{3.5}$$

Employing (2.4), equations (3.3) and (3.5) yield

$$\omega_n(k_n a_n)^{2(\nu-2)} \approx \beta_n, \quad \text{where} \quad \beta_n = P_n/N_n \quad (\equiv Q_n/E_n). \tag{3.6}$$

Here, β_n represents the ratio of the input flux, $\bar{p}(k)$, to the spectral density of the wave action, $\bar{N}(k)$, where the bar denotes averaging over the narrow spectral band between ω_n and ω_{n+1}. Assuming slow variation of the quantities involved, we shall drop the bar and use $\beta(\omega)$ in place of β_n. As with (2.6), we can rewrite (3.6) in terms of $S(\omega)$ and ω by using the scaling relationships (2.2), (2.7) and (2.8):

$$S(\omega) = \beta^{1/(\nu-2)}(\rho g^3)\,\omega^{(9-5\nu)/(\nu-2)}. \tag{3.7}$$

The exact equality used here implies that a non-dimensional constant of proportionality similar to the Kolmogorov constant has been introduced into the interaction coefficient β. This constant emerges in the empirical equation (3.10) below as the 'energy transfer coefficient' C_q.

The question remains as to the appropriate value of ν. The final, and most crucial, assumption of the present work, which is also consistent with the hypothesis of localness of the interaction process in (k,ω)-space, is that (2.9) remains valid even if Q is a (sufficiently smooth) function of ω. For such a non-conservative cascade, however, one must also accept that ν is a function of frequency: $\nu = \nu(\omega)$. This leads one to seek ν as an independent variable that makes (2.9) and (3.7) compatible.

The local energy flux $Q(\omega)$ through any given frequency ω is obtained by integrating the spectral density of the energy input from its lowest boundary to a given frequency ω (alternatively, to a given wavenumber k):

$$Q(\omega) = \int_0^k p(k)\,\omega\,\mathrm{d}k. \tag{3.8}$$

This flux can be tentatively broken down into two components:

$$Q(\omega) = Q_0 + Q_1(\omega), \quad \text{where} \quad Q_1(\omega) = \int_{\omega_0}^{\omega} \beta(\omega)\,S(\omega)\,\mathrm{d}\omega. \tag{3.9}$$

Here Q_0 is the energy flux through ω_0 and $Q_1(\omega)$ is the flux due to the Miles mechanism of wave generation. The wind–wave interaction coefficient $\beta(\omega)$ (the 'growth rate') (Miles 1962) rapidly decreases as ω approaches ω_0 from the right. The advantage of the breakdown (3.9) is that it allows one to take into account additional factors governing the wave spectrum. For instance, part of the total energy flux going to high frequencies (but generated at frequencies below ω_0) may be due to an interaction of the wave field with ocean current gradients, or due to the Phillips mechanism of wave excitation by a moving disturbance of atmospheric pressure; these can be included into Q_0. However, a much more important point is that ocean wave spectra usually contain an inverse cascade range whose extent depends on the wave age (associated with a shift of the spectral peak frequency ω_p to the left of the generation range ω_0). As a result, at frequencies below ω_0 the spectrum does not immediately drop to zero but continues to increase with decreasing ω. The spectral shape in this range can be approximated by (2.10) or (2.9) where $\nu_0 \rightarrow 4$ as $\omega \rightarrow \omega_p$. At $\omega \rightarrow \omega_0$ the spectrum should match (3.7). Hence, the boundary condition for the second term in (3.9) can be taken as $S(\omega_0) = S_0$, where S_0 is related to the 'net direct flux' Q_0 passing through ω_0. A rigorous determination of Q_0 would require a complete description of spectral fluxes including the inverse cascade of the wave action, which is well beyond the scope of the present work. To simplify the matter, we shall present our end results (§5) for several characteristic values of α. Its variation is equivalent to the effect of varying S_0 and/or Q_0, which can be demonstrated by selecting appropriate scales for all variables.

The commonly accepted form of $\beta(\omega)$ consistent with the Miles theory is

$$\beta(\omega) = \epsilon\omega\phi(\omega/\omega_0), \quad \text{where} \quad \omega_0 = g/U. \tag{3.10}$$

Here, $\epsilon = \rho_a/\rho$ is the ratio of the air and water densities ($\epsilon \sim 10^{-3}$) and U is the characteristic wind velocity well above the surface. The limit of $\epsilon \rightarrow 0$ corresponds to a conservative cascade, in which case (2.9) with $\nu = \text{const}$ applies also for $\omega > \omega_0$. The mean number of resonantly interacting components at $\omega > \omega_0$ is then found by solving (2.9) for ν at $\omega = \omega_0$, with S_0 and Q_0 assumed to be known. Hence, the rate of spectral roll-off at high frequencies is determined in this case by the conditions at low frequencies. An *a priori* acceptance of $\nu = 4$ or $\nu \rightarrow \infty$ is unwarranted.

The non-dimensional function ϕ in (3.10) increases monotonically, starting from about zero at $\omega = \omega_0$. Based on observations of rather poorly developed seas, Snyder *et al.* (1981) proposed:

$$\phi \approx \begin{cases} 0 & \text{for} \quad \omega \leqslant \omega_0 \\ C_q(\omega/\omega_0 - 1) & \text{for} \quad \omega > \omega_0. \end{cases} \tag{3.11}$$

An empirical constant C_q plays the role of the energy transfer coefficient ($C_q \sim 10^{-1}$). An alternative, quadratic form of $\phi(\omega)$ believed to be appropriate for larger values of the wave age was proposed by Plant (1982). In the calculations below both forms will be tested. However, since we do not consider effects of the inverse cascade, the value of the transfer coefficient C_q in our $\phi(\omega)$ must differ from the measured values. Such a difference is because the actually observed wind input is shared by a variety of wave–wave and wave–current interaction mechanisms, whereas the input implied in the present work is entirely channelled towards high wavenumbers in the inertial cascade.

4. Multi-wave interactions: an implicit solution

Equations (2.9), (3.7) and (3.8) yield a closed system of one integral and two algebraic equations for S, Q and ν. Let us carry out a preliminary analysis of these equations to evaluate the possible range of variation of the mean number ν of the resonantly interacting components. To this end we shall scale the variables in such

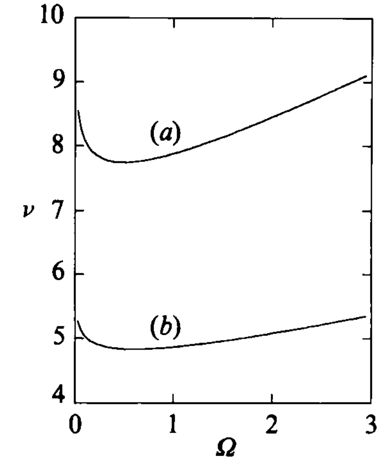

FIGURE 1. The number, $\nu(\omega)$, of resonantly interacting harmonics versus the natural logarithm of the non-dimensional frequency, as given by (4.5*a*) for $\phi(\omega) = C_q(\omega-1)$: (*a*) $\alpha_q = 0.1$ and $C_q = 0.01$; (*b*) $\alpha_q = 0.01$ and $C_q = 0.01$.

a way as to eliminate Q and thus reduce the problem to a system of two algebraic equations. This is achieved by setting:

$$\omega = (\rho g^3/Q)^{\frac{1}{3}}\tilde{\omega}, \quad S(\omega) = (Q^{\frac{5}{2}}/\rho g^3)^{\frac{2}{3}}\tilde{S}, \quad \beta = (\rho g^3/Q)^{\frac{1}{3}}\tilde{\beta}.$$

The relationship between $\tilde{\omega}$ and the true frequency ω will remain ambiguous until the dependence of Q on ω is resolved. However, comparisons with the explicit solution obtained in the following section and in the Appendix show that (4.5) below bring out major features of the spectrum behaviour. In a special case considered in the Appendix, the present approach yields explicit results which demonstrate that for most practical purposes, the frequency scale $\omega_* = (\rho g^3/Q)^{\frac{1}{3}}$ appearing in the above relationships can be identified with $\omega_0 = g/U$.

Hereinafter we shall use only the scaled quantities, although the tilde will be omitted. Equations (2.9) and (3.7) become

$$S(\omega) = \alpha\omega^{(8-5\nu)/(\nu-1)}, \tag{4.1}$$

$$S(\omega) = [\epsilon\phi(\omega)]^{1/(\nu-2)}\omega^{-5}. \tag{4.2}$$

Let us recall that (4.1) relates the shape of the wave spectrum (quantitied by ν) to the degree of nonlinearity (quantitied by $k_n a_n$), and (4.2) provides for the continuity of the action flux through the spectrum, given a variable external input. In logarithmic form these equations are

$$Z = A + \Omega(8-5\nu)/(\nu-1), \quad Z = -5\Omega + \Phi/(\nu-2) \tag{4.3}$$

where
$$Z = \ln S, \quad \Omega = \ln\omega, \quad A = \ln\alpha, \quad \Phi = \ln\epsilon\phi. \tag{4.4}$$

The (physically meaningful) solution of (4.3) appropriate for $\alpha < 1$ takes the form

$$\nu = (1/2A)(3A - 3\Omega + \Phi - [12A\Omega + (\Phi + A - 3\Omega)^2]^{\frac{1}{2}}), \tag{4.5a}$$

$$Z = \tfrac{1}{2}(A - 7\Omega - \Phi - [12A\Omega + (\Phi + A - 3\Omega)^2]^{\frac{1}{2}}). \tag{4.5b}$$

In figures 1 and 2, (4.5*a*, *b*) are illustrated for the case of $\phi(\omega) = C_q(\omega-1)$ corresponding to (3.11). One shortcoming of the function $\phi(\omega)$ proposed by Snyder *et al.* (1981) is the presence of a singularity in the vicinity of 1. The Phillips (1958) and Zakharov–Filonenko (1966) spectra are also plotted, for comparison.

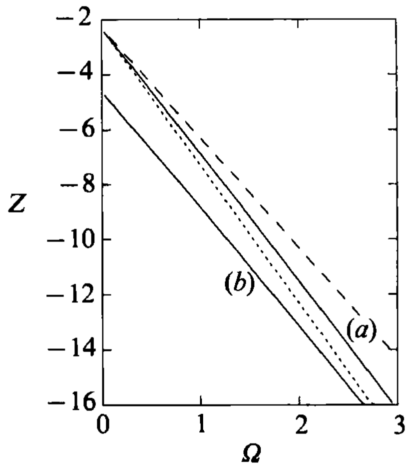

FIGURE 2. ——, The natural logarithm, Z, of the non-dimensional spectrum versus the natural logarithm of the non-dimensional frequency, as given by (4.5*b*), corresponding to figure 1; ....., the Phillips spectrum $Z = A - 5\Omega$; ----, the Zakharov–Filonenko spectrum $Z = A - 4\Omega$.

5. The explicit solution

Let us non-dimensionalize (2.9), (3.7) and (3.9) by selecting the following unambiguous relationships between original and scaled variables:

$$\omega = \omega_0 \tilde{\omega}, \quad S = \rho g^3 \omega_0^{-5} \tilde{S}, \quad \beta = \epsilon \omega_0 \tilde{\omega} \phi(\tilde{\omega}),$$
$$Q_1 = \epsilon \rho g^3 \omega_0^{-3} \tilde{Q}_1, \quad Q_0 = \rho g^3 \omega_0^{-3} \tilde{Q}_0,$$

where $\omega_0 = g/U$. For $\omega > \omega_0$ this yields

$$S(\omega) = \alpha(\epsilon Q_1)^{1/(\nu-1)} \omega^{(8-5\nu)/(\nu-1)}, \tag{5.1}$$

$$S(\omega) = (\epsilon\phi)^{1/(\nu-2)} \omega^{-5}, \tag{5.2}$$

$$Q_1 = \int_1^{\omega} \phi(\omega) S(\omega)\, \omega \, \mathrm{d}\omega. \tag{5.3}$$

Again, the tilde over the variables is implied but not written. The boundary condition takes the form

$$S(1) = \alpha Q_0^{1/(\nu_0 - 1)}. \tag{5.4}$$

Considering (5.1) and (5.2) as a system of algebraic equations one can solve it with respect to ν and Q_1 to obtain an integral Volterra equation for $S(\omega)$. Let us introduce 'the degree of saturation' employed earlier by Phillips (1985),

$$Y = S\omega^5, \tag{5.5}$$

and use the functions $\Phi = \ln(\epsilon\phi)$ and $\Psi = \ln Y$. Then, the solution of (5.1) and (5.2) can be written in a convenient form:

$$\alpha^{-1}\epsilon Q_1 = \exp(\Psi + \Phi)\, \omega^{-3} \alpha^{-\nu}, \tag{5.6}$$

$$\nu = 2 + \Phi/\Psi. \tag{5.7}$$

Differentiating (5.6) with respect to ω and changing to the logarithmic variables, Φ, Ψ, $A = \ln\alpha$ and $\Omega = \ln\omega$, we arrive ultimately at the following differential equation for Ψ:

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\Omega} = \frac{\Psi[A(\mathrm{d}\Phi/\mathrm{d}\Omega) + (\alpha^{-1+\nu} + 3 - (\mathrm{d}\Phi/\mathrm{d}\Omega))\,\Psi]}{\Psi^2 + A\Phi}. \tag{5.8}$$

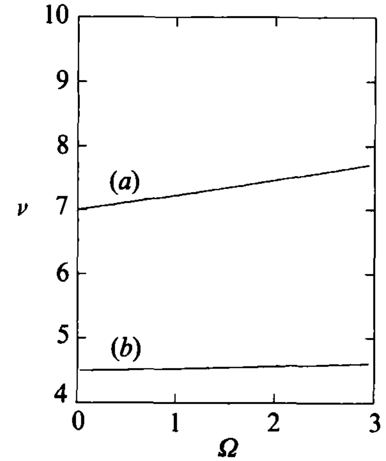

FIGURE 3. The number, $\nu(\omega)$, of resonantly interacting harmonics versus the natural logarithm of the non-dimensional frequency, for $\phi(\omega) = C_q \omega^2$, as based on the numerical solution of (6.9) with the boundary condition (5.9): (*a*) $\alpha_q = 0.1$ and $C_q = 0.01$; (*b*) $\alpha_q = 0.01$ and $C_q = 0.01$.

For $\alpha \ll 1$, the term $\alpha^{-1+\nu}$ appearing in the numerator can be neglected. The boundary condition for Ψ follows from (5.4) in which, without loss of generality, one can set $Q_0 = 1$. Therefore,

$$\Psi_0 = A. \tag{5.9}$$

Obtaining a numerical solution for any choice of $\Phi(\Omega)$ presents no difficulty. The calculation for the case of $\Phi(\Omega)$ based on (3.11) yields plots virtually identical to figure 1 and 2 in which ω_* is replaced by ω_0. In figure 3 we illustrate $\nu(\omega)$ for the case of $\phi(\omega) = C_q \omega^2$. The corresponding spectrum $S(\omega)$ is close to that shown in figure 2. In the next section and in the Appendix this case is also studied analytically for special values of external parameters.

6. Special cases

The linear dependence of ϕ on ω was discovered by Snyder *et al.* (1981) for rather poorly developed seas characterized by a strong wind–wave coupling. A simple analytical solution for this situation can be obtained by approximating

$$\phi = C_q \omega, \quad \text{hence} \quad \Phi(\Omega) = E + \Omega, \tag{6.1}$$

where $E = \ln(\epsilon C_q)$. Then, (5.8) reduces to

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\Omega} = \frac{\Psi(A+2\Psi)}{\Psi^2 + A(E+\Omega)}. \tag{6.2}$$

Field observations (Donelan, Hamilton & Hui 1985) indicate that α decreases with increasing wave age (alternatively, with the non-dimensional wind fetch). Equation (6.2) can be solved using a Taylor series expansion:

$$\Psi(\Omega) = \Psi_0 + a_1 \Omega + a_2 \Omega^2 + a_3 \Omega^3 \ldots. \tag{6.3}$$

Selecting $\Psi_0 = A$, the first three coefficients a_n are found to be

$$a_1 = \frac{3A}{A+E}, \quad a_2 = -\frac{3A(A-2E)}{(A+E)^3}, \quad a_3 = \frac{6(2A-E)(A-2E)}{(A+E)^5}. \tag{6.4}$$

It can be shown that for $n \geqslant 2$, all a_n contain the factor $(A-2E)$. The corresponding dimensionless spectrum can now be written in an instructive form:

$$S(\omega) \approx \alpha\omega^{-5+4\mu-\delta(\omega)}, \tag{6.5}$$

where

$$\mu = \tfrac{1}{4}a_1 \equiv 3A/[4(A+E)] \equiv \ln\alpha^3/\ln[(\alpha\epsilon C_q)^4], \quad \delta(\omega) \approx -(a_2+a_3\ln\omega+\ldots)\ln\omega. \tag{6.6}$$

The parameter μ appearing in the exponent of (6.5) has a simple geometrical interpretation (Glazman *et al.* 1988; Glazman & Weichman 1989) based on the fact that the Hausdorff (fractal) dimension of a Gaussian random surface characterized by the wavenumber spectrum $k^{-4+2\mu}$ is $D_{\mathrm{H}} = 2+\mu$. (The quantity μ is sometimes called the fractal codimension, although a more standard definition of the codimension is: $c = 3-D_{\mathrm{H}}$ where 3 is the dimension of the embedding space.). The effect of $\delta(\omega)$ on the spectrum shape is to increase the rate of spectrum roll-off above that given by $\omega^{-5+4\mu}$. However, owing to the smallness of a_n with $n \geqslant 2$, this effect becomes noticeable only for large ω. In terms of the original, dimensional, variables the spectrum takes the form

$$S(\omega) = \alpha\rho g^3(U/g)^{4\mu}\omega^{-5+4\mu-\delta(\omega U/g)}. \tag{6.7}$$

Apparently, at a given frequency ω, the effect of $\delta(\omega/\omega_0)$ becomes the more appreciable, the greater the wind. This form of the wave spectrum – with the wave-age-dependent μ and α—has been successfully employed for the explanation of various wave-age-related biases in satellite remote sensing measurements (e.g. Glazman 1990; Glazman & Srokosz 1991). However, in the previous work, effects of the wave age have been quantified using *ad hoc* principles rather than analysis of wave dynamics.

Another case of special interest is the quadratic law

$$\phi = C_q\omega^2, \quad \text{hence} \quad \Phi(\Omega) = E+2\Omega \tag{6.8}$$

which agrees better with observations at higher degrees of wave development (Plant 1982). Equation (5.8) takes the form

$$\frac{\mathrm{d}\Psi}{\mathrm{d}\Omega} = \frac{\Psi(2A+\Psi)}{\Psi^2+A(E+2\Omega)}. \tag{6.9}$$

It is easy to check that this equation is satisfied by a function

$$\Psi(\Omega) = \tfrac{1}{2}E+\Omega \tag{6.10}$$

which corresponds to

$$S(\omega) = (\epsilon C_q)^{\frac{1}{3}}\omega^{-4}. \tag{6.11}$$

According to (4.1), this spectrum is pertinent to four-wave interactions. Indeed, substituting (6.10) and (6.8) into (5.7) confirms that $\nu = 4$. Comparing (6.10) with (5.9) we see that $A = \frac{1}{2}E$. In other words, the regime of four-wave interactions is preserved if the energy flux Q_0 arriving at ω_0 from the low-frequency range does not exceed a certain, rather small, magnitude. In the Appendix this regime is explained in greater detail. In terms of the dimensional variables this spectrum is given by

$$S(\omega) = (\epsilon C_q)^{\frac{1}{3}}\rho g^3(U/g)\,\omega^{-4}. \tag{6.12}$$

Therefore, the Zakharov–Filonenko spectrum, which was originally derived on the assumption of a purely conservative cascade, remains valid in a special non-conservative situation. Earlier, Phillips (1985) arrived at the same conclusion, although $\nu = 4$ was assumed *a priori*.

The analysis of (2.9), (3.7) and (3.9) offered above is rather preliminary. A deeper understanding of these equations requires a better knowledge of the parameters α, ϕ and C_q treated here as external factors which depend on the wave age in a yet unspecified manner. Such knowledge is presently very poor.

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Appendix

Employing the approach presented in §4 one can obtain an explicit solution of (2.9), (3.7) and (3.9) for the special case $\phi(\omega) \sim \omega^2$. Although this case was treated in §6 based on an approximate version of (5.8), its re-examination with the exact equations of §4 is of intrinsic interest. (In this section all equations are written in terms of actual dimensional variables.)

Note that the function

$$\Phi(\Omega) = 2(A+\Omega) \tag{A 1}$$

generates two formal solutions to (4.3). The first solution is $\nu_1 = 4$ and $Z_1 = A - 4\Omega$. The second solution, $\nu_2 = A - \Omega/A$ and $Z_2 = -2A - 5\Omega$, would lead us to the Phillips spectrum. This solution is not considered in the present paper. In the dimensional variables the first solution takes the form of (1.3) where Q remains an unknown function of the frequency. Substituting this formal result into (3.9) and differentiating the resultant integral equation over ω we arrive at a differential equation for $Q(\omega)$:

$$\mathrm{d}Q(\omega)/\mathrm{d}\omega = \alpha(\rho g^3)^{\frac{2}{3}}\beta(\omega)\,[Q(\omega)]^{\frac{1}{3}}\omega^{-4}. \tag{A 2}$$

The precise dimensional form of $\beta(\omega)$ equivalent to (A 1), accounting for the scaling relationships of §4, is given by

$$\beta(\omega) = \alpha^2\omega(\omega/\omega_*)^2, \quad \text{where} \quad \omega_* = (\rho g^3/Q)^{\frac{1}{3}}. \tag{A 3}$$

Evidently, our choice of $\Phi(\Omega)$ is not quite equivalent to the quadratic law for $\phi(\omega)$ because $Q = Q(\omega)$. The quadratic law would result in

$$\beta(\omega) = \epsilon C_q\,\omega(\omega/\omega_0)^2. \tag{A 4}$$

However, it is easy to show that replacing ω_* with ω_0 in (A 3) leads to the desired result with sufficient accuracy. Let us insert (A 3) into (A 2) and integrate from ω_0 to ω. This yields

$$Q(\omega) = (\omega/\omega_0)^{\alpha^3}Q_0 \approx Q_0. \tag{A 5}$$

Since $\alpha \ll 1$, the last approximation remains accurate even at large ω. Therefore, for approximate calculations of the explicit dependence of S on ω starting from (4.5), the variation of the energy flux in ω_* can be discarded. Finally, selecting $\alpha^2 = \epsilon C_q$ makes (A 1) fully equivalent to (A 4), and α is indeed sufficiently small (for $C_q \sim 10^{-1}$, $\alpha \sim 10^{-2}$).

Apparently, having decreased the Kolmogorov constant (alternatively, the flux Q_0 arriving from the low-frequency range) to a certain – rather small – value, we have achieved the regime under which the local spectral energy at any $\omega > \omega_0$ satisfies the criterion of weak nonlinear interactions. The corresponding energy level is determined by (3.6) in which $\nu = 4$.

The regime resulting in (6.10) is likely to be realized at high degrees of sea maturity – when the inverse cascade is well developed and the advective flow of the wave energy is also large due to the large group velocity of long waves. When the wave age $C_p/U > 1$, the wind input is partitioned between the direct and the inverse cascades. As a result, one can anticipate that the direct flux remains relatively small even at a high wind speed, so that regime (6.12) can be realized.

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